An exact soliton solution for an averaged dispersion-managed fibre system equation



2001 J. Phys. A: Math. Gen. 34 5111

(http://iopscience.iop.org/0305-4470/34/24/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 5111–5115 www.iop.org/Journals/ja PII: S0305-4470(01)22914-X

# An exact soliton solution for an averaged dispersion-managed fibre system equation

**K Nakkeeran**

Laboratoire de physique de l'Université de Bourgogne, UMR CNRS No 5027, Av. A Savary, BP 47 870, 21078 Dijon Cédex, France

E-mail: naks@jupiter.u-bourgogne.fr

Received 9 March 2001, in final form 27 April 2001

**Abstract**

We consider the nonlinear wave propagation in an averaged dispersion-managed (DM) fibre system. We present the explicit Lax pair with a variable spectral parameter and derive the exact soliton solution using the Bäcklund transformation. A similar study is also carried out for simultaneous propagation of $N$ nonlinear pulses in the averaged DM fibre system.

PACS numbers: 42.81.Dp, 42.65.Tg, 05.45.Yu

Nonlinear pulse propagation in a long-distance, high-speed optical fibre transmission system can be described by the (perturbed) nonlinear Schrödinger (NLS) equation. The NLS equation includes the linear effect due to the group velocity of the pulse and the nonlinear effect due to the Kerr effect [1]. Many research works on the development of such a system have concentrated efforts on overcoming or controlling these effects [2, 3]. In this direction, recent numerical studies [4–6] and experiments [7] have shown that a periodic dispersion compensation seems to be the most effective way of improving the optical transmission system. The main purpose of the dispersion management is to reduce several effects such as radiation due to lumped amplifiers compensating the fibre loss [8, 9], modulational instability [10], jitters caused by the collisions between signals [11], and the Gordon–Haus effect resulting from the interaction with noise [12], and also to set a desired average value of the dispersion [10].

Basically, the dispersion-management technique utilizes a transmission line with a periodic dispersion map, such that each period is built up by two types of fibre, generally with different lengths and opposite group-velocity dispersion (GVD) [4]. Lakoba has proved the non-integrability of the system equation governing the pulse propagation in dispersion-managed (DM) fibres [13]. As there is no available analytical solution for DM solitons, to date researchers have utilized the Lagrangian method to efficiently study the dynamics of DM solitons [4]. Very recently we have developed a complete collective variable theory for DM solitons which effectively includes the residual field due to soliton dressing and radiation [14]. Many works have been reported to fit a Hermite–Gaussian ansatz function for the oscillating tails of the numerical stationary solution (fixed point) of the DM solitons [4, 15–17]. It was pointed out in numerical studies [5, 6] that in a DM fibre line the pulse is deformed from the ideal soliton, has a chirp and requires an enhanced power for the average dispersion. Meanwhile Kumar and Hasegawa [18] have obtained a new nonlinear pulse (quasi-soliton) by programming the dispersion profile such that the wave equation has a combination of the usual quadratic potential and the linear parabolic potential.

In this paper, we consider the DM soliton equation averaged over one dispersion map. We present the Lax pair and derive the exact soliton solution using the Bäcklund transformation. Finally we also present the Lax pair and derive the exact soliton solutions for the $N$ fields propagation.

Nonlinear pulse dynamics in a DM fibre is governed by the NLS equation

$$\mathrm{i}u_z + \frac{D(z)}{2}u_{\tau\tau} + \gamma|u|^2u = 0 \tag{1}$$

where $u$ represents the complex envelope amplitude, subscripts $\tau$ and $z$ denote the partial derivatives with respect to time and distance along the direction of propagation. $D(z)$ is the GVD parameter which periodically changes between normal and anomalous for the dispersion management and $\gamma$ is the parameter related to Kerr nonlinearity. Note that in equation (1), optical losses are not included.

Using a chirped ansatz function for equation (1), Hasegawa *et al* [19], have averaged it over one dispersion map and derived

$$\mathrm{i}u_z + \frac{D_0}{2}u_{\tau\tau} + \gamma_0|u|^2u + \kappa_0\tau^2u = 0 \tag{2}$$

where $D_0$, $\gamma_0$ and $\kappa_0$ are related to the averaged fibre and pulse parameters (see [19] for more details). Kumar and Hasegawa derived a chirped stationary solution of equation (2) [18].

Using the scaling $\tau = \sqrt{D_0/2}\,t$ and $u = \sqrt{2/\gamma_0}\,q$, equation (2) can be transformed to

$$\mathrm{i}q_z + q_{tt} + 2|q|^2q + \frac{\kappa_0 D_0}{2}t^2q = 0. \tag{3}$$

Now, consider the optical losses with loss parameter $\beta$ and $\kappa_0 D_0/2 = \beta^2$, so that equation (3) becomes

$$\mathrm{i}q_z + q_{tt} + 2|q|^2q + \beta^2t^2q + \mathrm{i}\beta q = 0. \tag{4}$$

Thus equation (4) becomes a special case of the averaged DM soliton equation (2) with optical losses.

The lax pair associated with equation (4) is derived as

$$\begin{aligned}\frac{\partial\Psi}{\partial t} &= U_1\Psi\\ \Psi &= (\psi_1\psi_2)^T\end{aligned} \tag{5}$$

where

$$U_1 = \begin{pmatrix} -\mathrm{i}\lambda & Q \\ -Q^* & \mathrm{i}\lambda \end{pmatrix} \tag{6}$$

where $Q = q\exp(-\mathrm{i}\beta t^2/2)$ and $\lambda$ is the variable spectral parameter given by

$$\lambda = \alpha(z) + \mathrm{i}\zeta(z) \qquad \lambda_z = -2\beta\lambda \qquad \lambda_t = 0 \tag{7a}$$

$$\lambda = \mu\exp(-2\beta z) \qquad \alpha(z) = \mathbb{R}(\mu)\exp(-2\beta z) \qquad \zeta(z) = \mathbb{I}(\mu)\exp(-2\beta z). \tag{7b}$$

Here $\mathbb{R}(\mu)$ and $\mathbb{I}(\mu)$, are respectively, the real and imaginary parts of the hidden iso-spectral parameter $\mu$. Space evolution of eigenfunction $\Psi$ is given by

$$\frac{\partial \Psi}{\partial z} = V_1 \Psi \tag{8}$$

$$V_1 = 2\mathrm{i}\lambda^2 \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + 2\lambda \begin{pmatrix} \mathrm{i}\beta t & Q \\ -Q^* & -\mathrm{i}\beta t \end{pmatrix} + \mathrm{i} \begin{pmatrix} |Q|^2 & Q_t + 2\mathrm{i}\beta t Q \\ Q_t^* - 2\mathrm{i}\beta t Q^* & -|Q|^2 \end{pmatrix}. \tag{9}$$

Equation (4) can be obtained from the compatibility condition $U_{1z} - V_{1t} + [U_1, V_1] = 0$.

We derive the Bäcklund transformation from the time evolution equation of the eigenfunction. In order to construct the Bäcklund transformation, let us write down equation (5) in terms of the Riccati equation. For this purpose, we introduce a new variable (or pseudopotential)

$$\Gamma = \frac{\psi_1}{\psi_2}. \tag{10}$$

Equation (10) yields,

$$\Gamma_t = -2\mathrm{i}\lambda\Gamma + Q + Q^*\Gamma^2. \tag{11}$$

Now transformation of variables $\Gamma \to \Gamma'$, $\lambda \to \lambda'$ and $Q \to Q'$ which keep the form of equation (11) invariant are sought. The simplest transformation can be tried by setting $\Gamma' = \Gamma$, $\lambda' = \lambda^*$ and looking for $Q'$ in the form

$$Q - Q' = \frac{2\mathrm{i}(\lambda - \lambda^*)\Gamma}{1 + |\Gamma|^2}. \tag{12}$$

Equation (12) defines the Bäcklund transformation of equation (4) with $Q = q \exp(-\mathrm{i}\beta t^2/2)$. Here the primed quantities refer to $N$-soliton solution and the unprimed quantities refer to $(N-1)$ soliton solution. To construct the soliton solution of equation (4), we start with the zero-soliton solution $Q = 0$. By substituting this trivial solution in equations (5) and (8), the explicit form of $\Gamma(0)$ is obtained as

$$\Gamma(0) = \exp[\xi(z,t) + \mathrm{i}\theta(z,t)] \tag{13}$$

where $\xi(z,t)$ and $\theta(z,t)$ are given by

$$\xi(z,t) = 2\zeta t + 8\int \alpha\zeta \,\mathrm{d}z - 4\beta t \int \zeta \,\mathrm{d}z \tag{14}$$

$$\theta(z,t) = -2\alpha t - 4\int (\alpha^2 - \zeta^2)\,\mathrm{d}z + 4\beta t \int \alpha \,\mathrm{d}z. \tag{15}$$

The explicit form of $\xi$ and $\theta$ can be respectively derived from equations (14) and (15) using equations (7). The one-soliton solution for equation (4) from (13) and with $q = Q\exp(\mathrm{i}\beta t^2/2)$ is derived as

$$q(z,t) = 2\zeta\,\mathrm{sech}(\xi)\exp(\mathrm{i}\theta + \mathrm{i}\beta t^2/2). \tag{16}$$

Thus we have derived the exact soliton solution for the averaged DM fibre system equation with losses using Bäcklund transformation. In [18], a chirped stationary soliton solution for equation (2) has been presented. Here, we have proved that the system equation (4) is completely integrable with the availability of the Lax pair and derived the exact soliton solution.

To achieve wavelength division multiplexing one needs to consider the simultaneous propagation of $N$ fields. When we consider the simultaneous propagation of $N$ fields in the averaged DM soliton equation (4), then the system equation can be written as

$$\mathrm{i}q_{jz} + q_{jtt} + 2q_j\left(\sum_{n=1}^{N} |q_n|^2\right) + \beta^2 t^2 q_j + \mathrm{i}\beta q_j = 0 \qquad j = 1, 2, \ldots, N. \tag{17}$$

The Lax pair associated with equation (17) is constructed as

$$\frac{\partial\Psi}{\partial t} = U_2\Psi \qquad \Psi = (\psi_1\psi_2\psi_3\cdots\psi_{N+1})^T \tag{18}$$

where

$$U_2 = \begin{pmatrix} -i\lambda & Q_1 & Q_2 & \cdots & Q_N \\ -Q_1^* & i\lambda & 0 & \cdots & 0 \\ -Q_2^* & 0 & i\lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_N^* & 0 & 0 & \cdots & i\lambda \end{pmatrix} \tag{19}$$

where $Q_j = q_j \exp(-i\beta t^2/2)$ and $\lambda$ is given by equations (7).

The space evolution of the eigenfunction $\Psi$ is given by

$$\frac{\partial\Psi}{\partial z} = V_2\Psi \tag{20}$$

$$V_2 = 2i\lambda^2 \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} + 2\lambda \begin{pmatrix} i\beta t & Q_1 & Q_2 & \cdots & Q_N \\ -Q_1^* & -i\beta t & 0 & \cdots & 0 \\ -Q_2^* & 0 & -i\beta t & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -Q_N^* & 0 & 0 & \cdots & -i\beta t \end{pmatrix}$$
$$+i \begin{pmatrix} \sum_{n=1}^N |Q_n|^2 & Q_{1t} + 2i\beta t Q_1 & Q_{2t} + 2i\beta t Q_2 & \cdots & Q_{Nt} + 2i\beta t Q_N \\ Q_{1t}^* - 2i\beta t Q_1^* & -|Q_1|^2 & -Q_2 Q_1^* & \cdots & -Q_N Q_1^* \\ Q_{2t}^* - 2i\beta t Q_2^* & -Q_1 Q_2^* & -|Q_2|^2 & \cdots & -Q_N Q_2^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ Q_{Nt}^* - 2i\beta t Q_N^* & -Q_1 Q_N^* & -Q_2 Q_N^* & \cdots & -|Q_N|^2 \end{pmatrix}. \tag{21}$$

System equation (17) can be obtained from the compatibility condition $U_{2z} - V_{2t} + [U_2, V_2] = 0$.

In order to construct the Bäckund transformation of equation (17), we introduce new variables (or pseudopotentials)

$$\Gamma_j = \frac{\psi_j}{\psi_{N+1}} \qquad j = 1, 2, \ldots, N. \tag{22}$$

Using the same procedure, the Bäcklund transformation for equation (17) is derived as

$$Q_j - Q_j' = \begin{cases} \dfrac{2i(\lambda - \lambda^*)\Gamma_1\Gamma_{j+1}^*}{1 + \sum_{n=1}^N |\Gamma_n|^2} & \text{for} \quad j = 1, 2, \ldots, N-1 \\ \dfrac{2i(\lambda - \lambda^*)\Gamma_1}{1 + \sum_{n=1}^N |\Gamma_n|^2} & \text{for} \quad j = N. \end{cases} \tag{23}$$

Similarly, the one-soliton solutions of equations (17) are generated as

$$q_j = \frac{2\zeta a_{j+1}^*}{a_1^*} \operatorname{sech}(\xi) \exp(i\theta + i\beta t^2/2) \qquad j = 1, 2, \ldots, N-1 \tag{24a}$$

$$q_N = \frac{2\zeta}{a_1^*} \operatorname{sech}(\xi) \exp(i\theta + i\beta t^2/2) \tag{24b}$$

with the condition $1 + \sum_{n=2}^N |a_n|^2 = |a_1|^2$.

In conclusion we have considered a special case of the averaged DM fibre system equation with fibre losses and presented the explicit Lax pair with variable spectral parameters. Using the Bäcklund transformation, we have successfully derived the exact soliton solution. A similar study is also extended for the simultaneous propagation of $N$ nonlinear pulses in the averaged DM fibre system.

## Acknowledgments

Financial support from the Centre National de la Recherche Scientifique (CNRS) is gratefully acknowledged. The author is also very grateful to S Wabnitz, P Tchofo Dinda, G Millot and A B Moubissi for their help.

## References

[1] Hasegawa A and Tappert F 1973 *Appl. Phys. Lett.* **23** 142
[2] Hasegawa A and Kodama Y 1995 *Solitons in Optical Communication* (New York: Oxford University Press)
[3] Agrawal G P 1989 *Nonlinear Fiber Optics* (San Diego: Academic)
[4] Zakharov V E and Wabnitz S 1998 *Optical Solitons: Theoretical Challenges and Industrial Perspectives* (Berlin: Springer)
[5] Smith N J, Knox F M, Doran N J, Blow K J and Bennion I 1996 *Electron. Lett.* **32** 54
[6] Georges T and Charbonnier B 1997 *IEEE Photonics Technol. Lett.* **9** 127
[7] Suzuki M, Morita I, Edagawa N, Yamamoto S, Toga H and Akiba S 1995 *Electron. Lett.* **31** 2027
[8] Forysiak W, Knox F M and Doran N J 1994 *J. Lightwave Technol.* **12** 1330
[9] Kumar S, Hasegawa A and Kodama Y 1997 *J. Opt. Soc. Am.* B **14** 888
[10] Doran N J, Smith N J, Forysiak W and Knox F M 1996 *Physics and Applications of Optical Solitons in Fibers '95* (Dordrecht: Kluwer)
[11] Hasegawa A, Kumar S and Kodama Y 1996 *Opt. Lett.* **21** 39
[12] Forysiak W, Blow K J and Doran N J 1993 *Electron. Lett.* **29** 1225
[13] Lakoba T I 1999 *Phys. Lett.* A **260** 68
[14] Tchofo Dinda P, Moubissi A B and Nakkeeran K 2001 *J. Phys. A: Math. Gen.* **34** L103
[15] Lakoba T I and Kaup D J 1998 *Phys. Rev.* E **58** 1998
[16] Turitsyn S K, Schäfer T, Spatschek K H and Mezentsev V K 1999 *Opt. Commun.* **163** 122
[17] Tchofo Dinda P, Nakkeeran K and Moubissi A B 2001 *Opt. Commun.* **187** 427
[18] Kumar S and Hasegawa A 1997 *Opt. Lett.* **22** 372
[19] Hasegawa A, Kodama Y and Maruta A 1997 *Opt. Fiber Technol.* **3** 197